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NOTES ON THE THEORY OF ASSOCIATION OF ATTRIBUTES IN STATISTICS.

BY G. UDNY YULE.

CONTENTS.

THE simplest possible form of statistical classification is "division" (as the logicians term it) "by dichotomy," i.e. the sorting of the objects or individuals observed into one or other of two mutually exclusive classes according as they do or do not possess some character or *attribute*; as one may divide men into sane and insane, the members of a species of plants into hairy and glabrous, or the members of a race of animals into males and females. The mere fact that we do employ such a classification in any case must not of course be held to imply a natural and clearly defined boundary between the two classes; e.g. sanity and insanity, hairiness and glabrousness, may pass into each other by such fine gradations that judgments may differ as to the class in which a given individual should be entered. The judgment must however be finally *decisive*; intermediates not being classed as such even when observed.

The theory of statistics of this kind is of a good deal of importance, not merely because they are of a fairly common type—the statistics of hybridisation experiments given by the followers of Mendel may be cited as recent examples—but because the ideas and conceptions required in such theory form a useful introduction to the more complex and less purely logical theory of variables. The classical writings on the subject are those of De Morgan*, Boole † and Jevons ‡, the method and notation of the latter being used in the following Notes, the first three sections of which are an abstract of the two memoirs referred to below §.

* *Formal Logic*, chap. VIII., "On the Numerically Definite Syllogism," 1847.

† *Analysis of Logic*, 1847. *Laws of Thought*, 1854.

‡ "On a General System of Numerically Definite Reasoning," *Memoirs of Manchester Literary and Philosophical Society*, 1870. Reprinted in *Pure Logic and other Minor Works*, Macmillan, 1890.

§ "On the Association of Attributes in Statistics," *Phil. Trans.* A, Vol. 194 (1900), p. 257. "On the theory of Consistence of Logical Class Frequencies," *Phil. Trans.* A, Vol. 197 (1901), p. 91.

1. *Notation; terminology; relations between the class frequencies; tabulation.*

The notation used is as follows*:

N	= total number of observations,		
(A)	= no. of objects or individuals	possessing attribute A,	
(α)	= "	"	*not* possessing attribute A,
(AB)	= "	"	possessing both attributes A and B,
$(A\beta)$	= "	"	" attribute A but not B.
(αB)	= "	"	" attribute B but not A,
$(\alpha\beta)$	= "	"	not possessing either attribute A or B,

and so on for as many attributes as are specified. A class specified by n attributes in this notation may be termed a class of the nth order. The attributes denoted by English capitals may be termed *positive* attributes, and their *contraries*, denoted by the Greek letters, *negative* attributes. If two classes are such that every attribute in the one is the negative or contrary of the corresponding attribute in the other they may be termed contrary classes, and their frequencies contrary frequencies; (AB) and $(\alpha\beta)$, $(AB\gamma)$ and $(\alpha\beta C)$ are for instance pairs of contraries.

If the complete series of frequencies arrived at by noting n attributes is being tabulated, frequencies of the same order should be kept together. Those of the same order are best arranged by taking separately the set or "aggregate" of frequencies, derivable from each positive class by substituting negatives for one or more of the positive attributes. Thus the frequencies for the case of three attributes may conveniently be tabulated in the order—

$$\left.\begin{array}{ll} \text{Order 0.} & N \\ \text{Order 1.} & (A), (\alpha) : (B), (\beta) : (C), (\gamma) \\ \text{Order 2.} & (AB), (A\beta), (\alpha B), (\alpha\beta) : (AC), (A\gamma), (\alpha C), (\alpha\gamma) : (BC), (B\gamma), \\ & (\beta C), (\beta\gamma) \\ \text{Order 3.} & (ABC), (\alpha BC), (A\beta C), (AB\gamma), (\alpha\beta C), (\alpha B\gamma), (A\beta\gamma), (\alpha\beta\gamma) \end{array}\right\} \quad (1).$$

But since all frequencies are used non-exclusively, (A) denoting the frequency of objects possessing the attribute A with or without others and so forth, the frequency of any class can always be expressed in terms of the frequencies of classes of higher order; that is to say we have

$$\left.\begin{aligned} N &= (A) + (\alpha) = (B) + (\beta) = \text{etc.} \\ &= (AB) + (A\beta) + (\alpha B) + (\alpha\beta) = \text{etc.} \\ (A) &= (AB) + (A\beta) \\ &= (ABC) + (AB\gamma) + (A\beta C) + (A\beta\gamma) = \text{etc.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

* I have substituted small Greek letters for Jevons' italics. Italics are rather troublesome when speaking, as one has to spell out a group like *AbcDE*, "big *A*, little *b*, little *c*, big *D*, big *E*." It is simpler to say $A\beta\gamma DE$. The Greek becomes more troublesome when many letters are wanted, owing to the non-correspondence of the alphabets, but this is not often of consequence.

Hence it is quite *unnecessary* to state *all* the frequencies as under (1); if space is of importance no more need be given than the eight frequencies of the third order, the *ultimate* frequencies as they may be termed (i.e. frequencies of classes specified by the whole number of attributes noted). The number of frequencies in an aggregate of order n, is evidently 2^n, so that not more than 2^n frequencies need be stated in any case where n attributes are observed.

Any other set of 2^n independent frequencies may however be chosen instead of the 2^n ultimate frequencies, the set formed by taking N together with the frequencies of all the positive classes offering several advantages. It is not difficult to see that any frequency whatever can be expressed in terms of the number of observations and the positive-class frequencies by using the relations (2). We have for instance

$$\left.\begin{aligned}(\alpha) &= N-(A)\\ (\alpha B) &= (B)-(AB)\\ (\alpha\beta) &= (\alpha)-(\alpha B) = N-(A)-(B)+(AB)\\ (\alpha\beta\gamma) &= (\alpha\beta)-(\alpha\beta C) = (\alpha\beta)-(\alpha C)+(\alpha BC)\\ &= N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)\end{aligned}\right\}\ldots\ldots(3),$$

and so on.

To take a very simple example with two attributes only, consider the statement of results of one of Mr Bateson's experiments on hybridisation with poultry, Leghorn-Dorking hybrids when crossed *inter se* produced offspring of varied forms; some having the rose comb and some not, some having the extra toe that characterises the Dorking and some not. Using A to denote "rose comb," B to denote extra toe, the numerical results may be completely expressed in either of the following forms

I	II
$(AB) = 208$	$N = 336$
$(A\beta) = 63$	$(A) = 271$
$(\alpha B) = 54$	$(B) = 262$
$(\alpha\beta) = 11$	$(AB) = 208.$

The advantages of the second form of tabulation are obvious; it gives at sight the whole number of observations and the numbers of A's and B's. The first table gives neither without reckoning, yet both are equally complete.

A rather interesting case arises where the frequencies of contrary classes are equal, as may be the case if the character dealt with is really variable and the points of division between A's and α's are taken at the medians. Such a condition implies necessary relations between the class-frequencies of any odd order and the frequencies of next lower order, but for the discussion of the case I must refer to the first memoir mentioned in my note § on p. 121.

2. *Consistence and Inference.*

Although the positive-class frequencies (including N under that heading) are all independent in the sense that no single one can be expressed in terms of the others, they are nevertheless subject to certain limiting conditions if they are to be self-consistent, i.e. such as might have been observed in one and the same field of observation or "universe," to use the convenient term of the logicians. Consider the case of three attributes, for example. It is evident that we must have

$$\left.\begin{array}{lll} (AB) \nless 0 & \text{as } (AB) & \text{must not be negative} \\ \quad \nless (A)+(B)-N & \text{as } (\alpha\beta) & \quad " \qquad " \\ \quad \ngtr (A) & \text{as } (A\beta) & \quad " \qquad " \\ \quad \ngtr (B) & \text{as } (\alpha B) & \quad " \qquad " \end{array}\right\} \ldots\ldots\ldots (4);$$

and similar conditions must hold for (AC) and (BC). But these are not the only conditions that must hold. The second-order frequencies must not only be such as not to imply negative values for the frequencies of other classes of their own aggregates, but also *must not imply negative values for any of the third-order frequencies.* Expanding all the third-order frequencies in terms of the frequencies of positive classes, and putting the resulting expansion $\nless 0$, we have

$$\left.\begin{array}{lr} & \text{or the frequency given below will be negative} \\ (ABC) \nless 0 & (ABC)\ [1] \\ \quad \nless (AB)+(AC)-(A) & (A\beta\gamma)\ [2] \\ \quad \nless (AB)+(BC)-(B) & (\alpha B\gamma)\ [3] \\ \quad \nless (AC)+(BC)-(C) & (\alpha\beta C)\ [4] \\ \quad \ngtr (AB) & (AB\gamma)\ [5] \\ \quad \ngtr (AC) & (A\beta C)\ [6] \\ \quad \ngtr (BC) & (\alpha BC)\ [7] \\ \quad \ngtr (AB)+(AC)+(BC)-(A)-(B)-(C)+N & (\alpha\beta\gamma)\ [8] \end{array}\right\} \ldots (5).$$

But if any one of the minor limits [1]—[4] be greater than any one of the major limits [5]—[8] these conditions are impossible of fulfilment. There are four minor limits to be compared with four major limits or sixteen comparisons in all to be made; but the majority of these, twelve in all, only lead back to conditions of the form (4). The four comparisons of expansions due to contrary frequencies alone lead to new conditions—viz.

$$\left.\begin{array}{r} (AB)+(AC)+(BC) \nless (A)+(B)+(C)-N \\ (AB)+(AC)-(BC) \ngtr (A) \\ (AB)-(AC)+(BC) \ngtr (B) \\ -(AB)+(AC)+(BC) \ngtr (C) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (6).$$

These conditions give limits to any one of the three frequencies (AB), (AC) and (BC) in terms of the other two and the frequencies of the first order, i.e. enable us to infer limits to the one class-frequency in terms of the others. It will very usually happen in practical statistical cases that the limits so obtained are valueless, lying outside those given by the simpler conditions (4), but that is merely because in practice the values of the assigned frequencies, e.g. (AB) and (AC), seldom approach sufficiently closely to their limiting values to render inference possible.

3. *Association.*

Two attributes, A and B, are usually defined to be independent, within any given field of observation or "universe," when the chance of finding them together is the product of the chances of finding either of them separately. The physical meaning of the definition seems rather clearer in a different form of statement, viz. if we define A and B to be independent *when the proportion of A's amongst the B's of the given universe is the same as in that universe at large.* If for instance the question were put "What is the test for independence of small-pox attack and vaccination?", the natural reply would be "The percentage of vaccinated amongst the attacked should be the same as in the general population" or "The percentage of attacked amongst the vaccinated should be the same as in the general population." The two definitions are of course identical in effect, and permit of the same simple symbolical expression in our notation; the criterion of independence of A and B is in fact

$$(AB)=\frac{(A)(B)}{N} \quad\text{..............................(7).}$$

In this equation the attributes specifying the universe are understood, not expressed. If all objects or individuals in the universe are to possess an attribute or series of attributes K it may be written

$$(ABK)=\frac{(AK)(BK)}{(K)}.$$

An equation of such form must be recognised as the criterion of independence for A and B within the universe K. As I have shewn in the first memoir referred to in note §, p. 121, if the relation (7) hold good, the three similar relations for the remaining frequencies of the "aggregate"—i.e. the set of frequencies obtained by substituting their contraries α, β for A or B or both—must also hold, viz.

$$\left.\begin{aligned}(\alpha B) &= \frac{(\alpha)(B)}{N}\\ (A\beta) &= \frac{(A)(\beta)}{N}\\ (\alpha\beta) &= \frac{(\alpha)(\beta)}{N}\end{aligned}\right\} \quad\text{..............................(8),}$$

for we have at once

$$(\alpha B) = (B) - (AB) = \frac{(B)\{N - (A)\}}{N} = \frac{(B)(\alpha)}{N},$$

and so on. The case of two attributes is thus quite simple; the definition of independence is almost intuitive, and the criterion need only be applied to one frequency of the aggregate. The two attributes are termed positively or negatively *associated* according as (AB) is greater or less than the value it would have in the case of independence, or, to put the same thing another way, according as $(AB)/(A)$ is greater or less than $(B)/N$ or $(AB)/(B)$ greater or less than $(A)/N$.

If more than two attributes are noted in the record they must be dealt with, in the first instance, pair by pair as above, but subsequently the association between each pair should be observed in the sections of material or sub-universes defined by the other attributes. In the case of three attributes, for instance, we have not only to deal with the association between A and B in the universe at large but also in the universe of C's and in the universe of γ's—associations which will be tested by comparing $(ABC)/(AC)$ with $(BC)/(C)$ or $(ABC)/(BC)$ with $(AC)/(C)$, and $(AB\gamma)/(A\gamma)$ with $(B\gamma)/(\gamma)$ and so on. Such "partial" associations are of great practical importance, as a test for the correctness or otherwise of physical interpretations placed on any "total" associations observed. When A and B are found to be associated it is a common form of argument to say that the association is not "direct" but due to the association of A with C and B with C; the argument may be tested at once by finding whether A and B are still associated in the separated universes C and γ. If they are the argument is baseless. It has been said for instance that the association between vaccination and protection from small-pox is due to the association of both with sanitary conditions (a larger proportion of the upper classes than of the lower classes being vaccinated, according to the argument). To test the argument the "universe" or material observed should be limited *either* to those living wholly in sanitary conditions *or* to those living wholly in insanitary conditions (it does not matter which). "Partial" associations again are of importance to the biologist in the theory of heredity. If an attribute be heritable, its presence in the parent or the grandparent is associated with its presence in the offspring; but the physical interpretation to be placed on such inheritance depends very largely on whether there is also a *partial* inheritance from the grandparent, the presence of the attribute in the grandparent being associated with its presence in the offspring even when the parents either all possess or all do not possess the attribute.

It is important to notice that the test for association is necessarily based on a *comparison* of percentages or proportions, e.g. $(AB)/(A)$ and $(B)/N$. The mere fact that a certain number of A's are B gives no physical information; besides knowing how many A's are B you must know also how many non-A's are B or what proportion of A's exists in the given universe at large. In an investigation as to the causation of A it is therefore just as important to observe non-A's as A's.

This point is frequently forgotten. In an investigation as to the inheritance of deaf-mutism in America*, for instance, only the offspring of deaf-mutes were observed, and the argument consequently breaks down on page after page into conjectural statements as to points on which the editor has no information—e.g. the proportion of deaf-mutes amongst the children of normals.

The difference of $(AB)/(A)$ from $(B)/N$ and of $(AB)/(B)$ from $(A)/N$ are of course not, as a rule, the same, and it would be useful and convenient to measure the "association" by some more symmetrical method—a "coefficient of association" ranging between ± 1 like the coefficient of correlation. In the first memoir referred to in note §, p. 121, such a coefficient, of empirical form, was suggested, but that portion of the memoir should now be read in connection with a later memoir by Professor Pearson †.

4. *On the theory of complete independence of a series of Attributes.*

The tests for independence are by no means simple when the number of attributes is more than two. Under what circumstances should we say that a series of attributes $ABCD\ldots$ were completely independent? I believe not a few statisticians would reply at once "if the chance of finding them together were equal to the product of the chances of finding them separately," yet such a reply would be in error. The mere result

$$\frac{(ABCD\ldots)}{N} = \frac{(A)}{N}\cdot\frac{(B)}{N}\cdot\frac{(C)}{N}\cdot\frac{(D)}{N}\cdots\cdots \qquad \ldots\ldots\ldots\ldots(9)$$

does not in general give any information as to the independence or otherwise of the attributes concerned. If the attributes are *known to be* completely independent then certainly the relation (9) holds good, but the converse is not true. "Equations of independence" of the form (9) must be shewn to hold for more than one class of any aggregate, of an order higher than the second, before the complete independence of the attributes can be inferred.

From the physical point of view complete independence can only be said to subsist for a series of attributes $ABCD\ldots$ within a given universe, when every pair of such attributes exhibits independence not only within the universe at large but also in every sub-universe specified by one or more of the remaining attributes of the series, or their contraries. Thus three attributes A, B, C are completely independent within a given universe if AB, AC and BC are independent within that universe and also

AB	independent	within the	universes	C and	γ,
AC	"	"	"	B "	β,
BC	"	"	"	A "	α.

* *Marriages of the Deaf in America*, ed. by E. A. Fay. Volta Bureau, Washington, 1898.
† *Phil. Trans.* Vol. 195, p. 16.

If a series of attributes are completely independent according to this definition relations of the form (9) must hold for the frequency of every class of every possible order. Take the class-frequency $(ABCD)$ of the fourth order for instance. A and B are, by the terms of the definition, independent within the universe CD. Therefore

$$(ABCD)=\frac{(ACD)(BCD)}{(CD)}.$$

But A and C, and also B and C, are independent within the universe D. Therefore the fraction on the right is equal to

$$\frac{1}{(CD)}\cdot\frac{(AD)(CD)}{(D)}\cdot\frac{(BD)(CD)}{(D)}=\frac{(AD)(BD)(CD)}{(D)^2}.$$

But again AD, BD, CD are each independent within the universe at large; therefore finally

$$(ABCD)=\frac{1}{(D)^2}\cdot\frac{(A)(D)}{N}\cdot\frac{(B)(D)}{N}\cdot\frac{(C)(D)}{N}=\frac{(A)(B)(C)(D)}{N^3}.$$

Any other frequency can be reduced step by step in precisely the same way.

Now consider the converse problem. The total frequency N is given and also the n frequencies (A), (B), (C), etc. In how many of the ultimate frequencies $(ABCD\ldots MN)$, $(\alpha BCD\ldots MN)$, etc. must "relations of independence" of the form

$$(ABCD\ldots MN)=\frac{(A)(B)(C)(D)\ldots(M)(N)}{N^{n-1}}$$

hold good, in order that complete independence of the attributes may be inferred? The answer is suggested at once by the following consideration. The number of ultimate frequencies (frequencies of order n) is 2^n; the number of frequencies given is $\overline{n+1}$. If then all but $\overline{n+1}$ of the ultimate frequencies are given in terms of the equations of independence, the remaining frequencies are determinate; either these determinate values must be those that would be given by equations of independence, *or a state of complete independence must be impossible.* Suppose all the ultimate class-frequencies to have been tested and found to be given by the equations of independence, with the exception of the negative class $(\alpha\beta\gamma\delta\ldots\mu\nu)$ and the n classes with one positive attribute $(A\beta\gamma\delta\ldots\mu\nu)$, $(\alpha B\gamma\delta\ldots\mu\nu)$, etc. Take any one of these untested class-frequencies, $(A\beta\gamma\delta\ldots\mu\nu)$, and we have for example

$$\begin{aligned}(A\beta\gamma\delta\ldots\mu\nu)=(A)&-(ABCD\ldots MN)\\ &-(ABCD\ldots M\nu)\\ &-\text{other terms with one negative}\\ &-(ABCD\ldots\mu\nu)\\ &-\text{other terms with two negatives}\\ &-\ldots\ldots\ldots\ldots\ldots\ldots\\ &-(AB\gamma\delta\ldots\mu\nu)\\ &-\text{other terms with } \overline{n-2} \text{ negatives.}\end{aligned}$$

But all the frequencies on the right are given by the relations of independence. Therefore

$$
\begin{aligned}
(A\beta\gamma\delta\ldots\mu\nu)=\frac{(A)}{N^{n-1}}\{N^{n-1} &- (B)(C)(D)\ldots(M)(N)\\
&- (B)(C)(D)\ldots(M)(\nu)\\
&- \ldots\ldots\ldots\ldots\\
&- (B)(C)(D)\ldots(\mu)(\nu)\\
&- \ldots\ldots\ldots\ldots\\
&- \ldots\ldots\ldots\ldots\\
&- (B)(\gamma)(\delta)\ldots(\mu)(\nu)\\
&- \ldots\ldots\ldots\ldots\}.
\end{aligned}
$$

Now consider the terms on the right in the bracket. With the exception of the one term $(B)(\gamma)(\delta)\ldots(\mu)(\nu)$, the remainder can be grouped in pairs of which the one member contains (B) and the other (β), the following frequencies in each member of the pair being the same. Carrying out this rearrangement the expression will read

$$
\begin{aligned}
(A\beta\gamma\delta\ldots\mu\nu)=\frac{(A)}{N^{n-1}}\{N^{n-1} &- (B)(\gamma)(\delta)\ldots(\mu)(\nu)\\
&- (B)(C)(D)\ldots(M)(N)\\
&- (\beta)(C)(D)\ldots(M)(N)\\
&- (B)(C)(D)\ldots(M)(\nu)\\
&- (\beta)(C)(D)\ldots(M)(\nu)\\
&- \ldots\ldots\ldots\ldots\\
&- (B)(C)(\delta)\ldots(\mu)(\nu)\\
&- (\beta)(C)(\delta)\ldots(\mu)(\nu)\\
&- \ldots\ldots\ldots\ldots\}.
\end{aligned}
$$

Replace (B) by $N-(\beta)$ throughout and rearrange the terms in similar pairs containing C and γ. (B) and (β) are then eliminated from all the terms and the expression then becomes

$$
\begin{aligned}
(A\beta\gamma\delta\ldots\mu\nu)=\frac{(A)}{N^{n-1}}\{N^{n-1} &+ (\beta)(\gamma)(\delta)\ldots(\mu)(\nu)\\
&- N(\gamma)(\delta)\ldots(\mu)(\nu)\\
&- N(C)(\delta)\ldots(\mu)(\nu)\\
&- \ldots\ldots\ldots\ldots\\
&- N(C)(D)\ldots(M)(N)\\
&- N(\gamma)(D)\ldots(M)(N)\\
&- \ldots\ldots\ldots\ldots\\
&- N(C)(D)\ldots(M)(\nu)\\
&- N(\gamma)(D)\ldots(M)(\nu)\\
&- \ldots\ldots\ldots\ldots\}.
\end{aligned}
$$

Replacing C by $N-(\gamma)$ and regrouping in similar pairs of terms containing (D) and (δ) this will become

$$(A\beta\gamma\delta \dots \mu\nu) = \frac{(A)}{N^{n-1}} \{N^{n-1} + (\beta)(\gamma)(\delta) \dots (\mu)(\nu) - N^{1}(D)(E) \dots (M)(N) - N^{1}(\delta)(E) \dots (M)(N) - \text{etc.}\}$$

and continuing the same process until all the frequencies $(D)(E) \dots (M)(N)$ are eliminated, i.e. $\overline{n-1}$ times altogether,

$$(A\beta\gamma\delta \dots \mu\nu) = \frac{(A)(\beta)(\gamma)(\delta) \dots (\mu)(\nu)}{N^{n-1}}.$$

That is to say the theorem must be true quite generally: "A series of n attributes $ABC \dots MN$ are completely independent if the relations of independence are *proved* to hold for $(2^n - \overline{n+1})$ of the 2^n ultimate frequencies; such relations must then hold for the remaining $\overline{n+1}$ frequencies also." If the ultimate frequencies are only given by the relations of independence in n cases or less, independence may exist for certain pairs of attributes in certain universes but not in general. The mere fact of the relation holding for one class, e.g.

$$(ABCD \dots MN) = \frac{(A)(B)(C)(D) \dots (M)(N)}{N^{n-1}},$$

implies nothing—in striking contrast to the simple case of two attributes, where $2^n - \overline{n+1} = 1$ and only the one class-frequency need be tested in order to see if independence exists. In the case of three attributes the number of third-order classes is eight, of which four must be tested in order to be certain that complete independence exists. In the case of four attributes there are sixteen fourth-order classes of which eleven must be tested, and so on.

I have dealt with the problem hitherto on the assumption that only the first-order and the nth order frequencies were given, and that the frequencies of intermediate orders were unknown—or at least uncalculated, for of course the frequencies of all lower orders may be expressed in terms of those of the nth order. If however the frequencies of all orders may be supposed known, the above result may be thrown into a somewhat interesting form. It will be remembered that the frequency of any class of any order may be expressed in terms of the frequencies of the *positive* classes $[(A)(AB)(AC)(ABC)$ etc.] of its own and lower orders. Then *complete independence* exists for a series of attributes *if the criterion of independence hold for all the positive-class frequencies up to that of the nth order*. If we have for instance

$$(ABCD \dots MN) = \frac{1}{N^{n-1}} \{(A)(B)(C)(D) \dots (M)(N)\},$$

and also

$$(BCD \dots MN) = \frac{1}{N^{n-2}} \{(B)(C)(D) \dots (M)(N)\},$$

we must have

$$\begin{aligned}(\alpha BCD \dots MN) &= (BCD \dots MN) - (ABCD \dots MN) \\ &= \frac{1}{N^{n-1}} \{(B)(C)(D) \dots (M)(N)\} \{N - (A)\} \\ &= \frac{1}{N^{n-1}} (\alpha)(B)(C)(D) \dots (M)(N),\end{aligned}$$

and so on. The number of class-frequencies to be tested in order to demonstrate the existence of complete independence is, of course, the same as before, viz. $2^n - \overline{n+1}$.

It should be noted as a consequence of these results that the definition of "complete independence" given on p. 127 is redundant in its terms. It is quite true that if complete independence subsist for a series of attributes every possible pair must exhibit independence in every possible sub-universe as well as in the universe at large, but it is not necessary to apply the criterion of independence to *all* these possible cases. In the case of three attributes for instance the criterion of independence need only be applied to four frequencies, as we have just seen, in order to demonstrate complete independence; it cannot then be *necessary*, as suggested by the definition, to test nine different associations, viz.

$$\begin{array}{lll} | AB | & | AB \mid C | & | AB \mid \gamma |, \\ | AC | & | AC \mid B | & | AC \mid \beta |, \\ | BC | & | BC \mid A | & | BC \mid \alpha |, \end{array}$$

in the notation of my memoir on Association (an expression like $| AB \mid C |$ specifying "the association between A and B in the universe of C's"). It is in fact only necessary to test $| AB |$, $| AC |$, $| BC |$, and $| AB \mid C |$ (or one of the other three partial associations in positive universes). If these are zero, the remaining associations must be zero also; for we are given

$$(ABC) = \frac{1}{(C)} (AC)(BC) = \frac{1}{N^2} (A)(B)(C),$$

$$\begin{aligned}\therefore (ABC) &= \frac{1}{(B)} (AB)(BC) \\ &= \frac{1}{(A)} (AB)(AC) = \text{etc.}\end{aligned}$$

i.e. $| AC \mid B |$, $| BC \mid A |$, etc. are zero. Quite generally, it is only necessary, if the testing be supposed to proceed from the second order classes upwards, to test *one* of all the possible partial associations corresponding to each positive class. If there be four attributes $ABCD$, the six total associations $| AB |$, $| AC |$, $| AD |$, $| BC |$

etc. must first be tried; if they are zero, then follow on with $|AB|C|$, $|AB|D|$, $|AC|D|$ and $|BC|D|$, but not $|AC|B|$ or $|AD|B|$ etc.; if they are zero then finally try $|AB|CD|$, if it also be zero then the attributes are completely independent. It is not necessary to try further $|AC|BD|$ or $|AD|BC|$ etc.

The inadequacy of the usual treatment of independence arises from the fact that it proceeds wholly *à priori*, and generally has reference solely to cases of artificial chance. The result is an illusory appearance of simplicity. It is pointed out that if one "event" can "succeed" in a_1 and "fail" in b_1 ways, a second succeed in a_2 and fail in b_2 ways, and so on, the combined events can take place (succeed or fail) in

$$(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)$$

ways and succeed in

$$a_1a_2\dots a_n$$

ways. The chance of entire "success" is therefore

$$\frac{a_1a_2\dots a_n}{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)},$$

the chance of the first event failing and the rest succeeding is

$$\frac{b_1a_2\dots a_n}{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)},$$

and so on for all other possible cases. In short the chance of occurrence of the combined independent events is the product of the chances of the separate events. There the treatment stops, and all practical difficulties are avoided. In such text-book treatment it is *given* that the events are independent and *required* to deduce the consequences; in the problems that the statistician has to handle the consequences—the bare facts—are given and it is required to find whether the "events" or attributes are independent, wholly or in part.

5. *On the fallacies that may be caused by the mixing of distinct records.*

It follows from the preceding work that we cannot infer independence of a pair of attributes within a sub-universe from the fact of independence within the universe at large. From $|AB|=0$, we cannot infer $|AB|C|=0$ or $|AB|\gamma|=0$, although we can of course make the corresponding inference in the case of complete association—i.e. from $|AB|=1$ we do infer $|AB|C|=|AB|\gamma|=\text{etc.}=1$. But the converse theorem is also true; a pair of attributes does not necessarily exhibit independence within the universe at large even if it exhibit independence in *every* sub-universe; given

$$|AB|C|=0, \qquad |AB|\gamma|=0,$$

we cannot infer $|AB|=0$. The theorem is of considerable practical importance from its inverse application; i.e. even if $|AB|$ have a sensible positive or negative value we cannot be sure that nevertheless $|AB|C|$ and $|AB|\gamma|$ are not both zero. Some given attribute might, for instance, be inherited neither in the male line nor the female line; yet a mixed record might exhibit a considerable apparent inheritance. Suppose for instance that 50 °/o of the fathers and of the sons exhibit the attribute, but only 10 °/o of the mothers and daughters. Then if there be no inheritance in either line of descent the record must give (approximately)

fathers	with	attribute	and sons	with	attribute	25 °/o
"	"	"	"	without	"	25 °/o
"	without	"	"	with	"	25 °/o
"	"	"	"	without	"	25 °/o

mothers	with	attribute	and	daughters	with	attribute	1 °/o
"	"	"	"	"	without	"	9 °/o
"	without	"	"	"	with	"	9 °/o
"	"	"	"	"	without	"	81 °/o.

If these two records be mixed in equal proportions we get

parents	with	attribute	and	offspring	with	attribute	13 °/o
"	"	"	"	"	without	"	17 °/o
"	without	"	"	"	with	"	17 °/o
"	"	"	"	"	without	"	53 °/o

Here $13/30 = 43\frac{1}{3}$ °/o of the offspring of parents with the attribute possess the attribute themselves, but only 30 °/o of offspring in general, i.e. there is quite a large but illusory inheritance created simply by the mixture of the two distinct records. A similar illusory association, that is to say an association to which the most obvious physical meaning must not be assigned, may very probably occur in any other case in which different records are pooled together or in which only one record is made of a lot of heterogeneous material.

Consider the case quite generally. Given that $|AB|C|$ and $|AB|\gamma|$ are both zero, find the value of (AB). From the data we have at once

$$(AB\gamma) = \frac{(A\gamma)(B\gamma)}{(\gamma)} = \frac{[(A)-(AC)][(B)-(BC)]}{[N-(C)]},$$

$$(ABC) = \frac{(AC)(BC)}{(C)}.$$

Adding

$$(AB) = \frac{N(AC)(BC)-(A)(C)(BC)-(B)(C)(AC)+(A)(B)(C)}{(C)[N-(C)]}.$$

Write

$$(AB)_0 = \frac{1}{N}(A)(B), \quad (AC)_0 = \frac{1}{N}(A)(C), \quad (BC)_0 = \frac{1}{N}(B)(C),$$

subtract $(AB)_0$ from both sides of the above equation, simplify, and we have

$$(AB) - (AB)_0 = \frac{N[(AC) - (AC)_0][(BC) - (BC)_0]}{C[N - (C)]}.$$

That is to say, *there will be apparent association between A and B in the universe at large unless either A or B is independent of C.* Thus, in the imaginary case of inheritance given above, if A and B stand for the presence of the attribute in the parents and the offspring respectively, and C for the male sex, we find a positive association between A and B in the universe at large (the pooled results) because A and B are both positively associated with C, i.e. the males of both generations possess the attribute more frequently than the females. The "parents with attribute" are mostly males; as we have only noted offspring of the same sex as the parents, their offspring must be mostly males in the same proportion, and therefore more liable to the attribute than the mostly-female offspring of "parents without attribute." It follows obviously that if we had found no inheritance to exist in any one of the *four* possible lines of descent (male-male, male-female, female-male, and female-female), no fictitious inheritance could have been introduced by the pooling of the *four* records. The pooling of the two records for the crossed-sex lines would give rise to a fictitious negative inheritance—disinheritance—cancelling the positive inheritance created by the pooling of the records for the same-sex lines. I leave it to the reader to verify these statements by following out the arithmetical example just given should he so desire.

The fallacy might lead to seriously misleading results in several cases where mixtures of the two sexes occur. Suppose for instance experiments were being made with some new antitoxin on patients of both sexes. There would nearly always be a difference between the case-rates of mortality for the two. If the female cases terminated fatally with the greater frequency and the antitoxin were administered most often to the males, a *fictitious* association between "antitoxin" and "cure" would be created at once. The general expression for $(AB) - (AB)_0$ shews how it may be avoided; it is only necessary to *administer the antitoxin to the same proportion of patients of both sexes.* This should be kept constantly in mind as an essential rule in such experiments if it is desired to make the most use of the results.

The fictitious association caused by mixing records finds its counterpart in the spurious correlation to which the same process may give rise in the case of continuous variables, a case to which attention was drawn and which was fully discussed by Professor Pearson in a recent memoir*. If two separate records, for each of which the correlation is zero, be pooled together, a spurious correlation will necessarily be created unless the mean of one of the variables, at least, be the same in the two cases.

* *Phil. Trans.* A, Vol. 192, p. 277.